Assignment 9.

This homework is due *Thursday*, November 1.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much. Extra problems (if there are any) are due December 7.

1. Quick reminder

(B) For a bounded function on a set E of finite measure, Lebesgue integral $\int_E f$ is defined as the common value of $\sup\{\int_E \varphi \mid \varphi \leq f, \varphi \text{ simple}\}$ and $\inf\{\int_E \psi \mid \psi \geq f, \psi \text{ simple}\}$, if the latter two are equal (which is guaranteed if f is measurable).

(P) Further, for an arbitrary *nonnegative* measurable function $f : E \to \mathbb{R} \cup \pm\infty$, define its Lebesgue integral by

$$\int_E f = \sup\left\{ \int_E h \;\middle|\; h \text{ bounded, measurable, of finite support and } 0 \leq h \leq f \text{ on } E \right\}$$

Both integrals defined above in (B) and (P) are linear, monotone and domain additive. Moreover, the following convergence theorem holds.

The Bounded Convergence Theorem. Let $\{f_n\}$ be a sequence of measurable functions on a set of finite measure E; let $\{f_n\}$ be uniformly bounded on E. If $\{f_n\} \to f$ pointwise on E, then

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

2. Exercises

(1) (4.1.6+) Prove that a continuous function on a closed interval $[a, b]$ is Riemann integrable. (*Hint:* Use uniform continuity to put an upper bound on the difference between upper and lower Darboux sums.)

(2) (4.2.9, 3.17) Let E have measure zero. Show that if f is a function on E, then f is measurable and $\int_E f = 0$
  - (a) for the definition (B) (assuming f is bounded),
  - (b) for the definition (P) (assuming nonnegative $f : E \to \mathbb{R} \cup \pm\infty$, including the case $f = \infty$ everywhere on E).

(3) (4.2.10+) Lef f be a measurable function on a set E. For a measurable subset A of E, show that $\int_A f = \int_E \chi_A f$
  - (a) for the definition (B) (assuming f is bounded and E is of finite measure),
  - (b) for the definition (P).

(4) (4.2.13) Show that the Bounded convergence theorem fails if we drop
  - (a) the assumption that the sequence $\{f_n\}$ is uniformly bounded,
  - (b) the assumption $m(E) < \infty$.

— see next page —

(5) Let f be a *semisimple* function, i.e. a function of the form $f = \sum_{n=1}^{\infty} \lambda_n \chi_{E_n}$ for some measurable sets E_n and real numbers λ_n. Assume additionally that f is bounded and of finite support. Prove that $\int_{\mathbb{R}} f = \sum_{n=1}^{\infty} \lambda_n m(E_n)$. (*Hint:* Use the Bounded convergence theorem.)

(6) (4.3.19) For a number $\alpha \in \mathbb{R}$, define $f(x) = x^{\alpha}$ for $0 < x \leq 1$ and $f(0) = 0$. Compute $\int_{[0,1]} f$. (*Hint:* In the bounded case, use connection to the Riemann integral. For the unbounded case, assume h in (P) is bounded by a number M. Using $h \leq f$, compute the largest possible value of $\int_{[0,1]} h$.)

(7) (4.3.18) Show that the definition (P) extends (B). That is, show that for a measurable bounded function on a finite measure set both definition give the same value of integral.